ISRAEL JOURNAL OF MATHEMATICS, Vol. 23, Nos. 3–4, 1976

GENERALIZED HEXAGONS OF ORDER (t, t)

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ABSTRACT

A generalized hexagon of order (t, t) in which certain subsets are maximal may be characterized as the generalized hexagon associated with Dickson's group $G_2(t)$. From this geometric result, it follows that if G is a group of automorphisms of a generalized hexagon of order (p, p) for a prime p and if G has rank 4 on points, then $G \trianglerighteq G_2(p)$.

1. Introduction

A finite *generalized hexagon* of order (s, t) [3] is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$, with an incidence relation satisfying the following axioms:

(i) each element of $\mathcal{P}$, which is called a point, is incident with $1 + t$ elements of $\mathcal{B}$, which are called edges, for $t \geqq 1$ and each edge is incident with $1 + s$ points for $s \geqq 1$;

(ii) $|\mathcal{P}| = (1 + s)(1 + st + s^2t^2)$ and $|\mathcal{B}| = (1 + t)(1 + st + s^2t^2)$;

(iii) 6 is the smallest positive integer k such that S has a circuit consisting of k points and k edges.

The distance of two elements $a, b \in \mathcal{P} \cup \mathcal{B}$ is denoted by $d(a, b)$ or $d(b, a)$. For $x \in \mathcal{P}$, let $x^\perp = \{y \in \mathcal{P}: d(x, y) \leqq 4\}$ and for distinct points x, y let $R(xy) = \bigcap \{z^\perp: z \in \mathcal{P} \text{ and } x, y \in z^\perp\}$.

We note that the only known examples of finite generalized hexagons are those associated with Dickson's group $G_2(q)$, which have order (q, q) where q is a prime power, and those associated with the triality group ${}^3D_4(q)$, which have order (q^3, q). We will study the case where $s = t$.

THEOREM 1.1. *Let $\mathcal{H}$ be a generalized hexagon of order (t, t). Suppose $|R(xy)| = 1 + t$ for all pairs (x, y) with $d(x, y) = 4$. Then $\mathcal{H}$ is isomorphic to the usual hexagon associated with $G_2(t)$.*

Received April 28, 1975

An important tool used for this geometric characterization is a theorem of Buekenhout and Shult [1] on pre-polar spaces. Indeed the incidence structure $(\mathcal{P}, \mathcal{L}, \in)$ where $\mathcal{L} = \mathcal{B} \cup \{R(xy): x, y, \in \mathcal{P}, d(x, y) = 4\}$ forms a pre-polar space. It follows that $\mathcal{P}$ together with its subspaces is the orthogonal geometry of dimension 7 over GF(t) and that t is a prime power. That $\mathcal{H}$ is the generalized hexagon associated with $G_2(t)$ is a consequence of work of Schellekens [5].

Next we study the case where $s = t$ is prime and Aut $\mathcal{H}$ has rank 4 on points.

THEOREM 1.2. *Let $\mathcal{H}$ be a generalized hexagon of order (p, p) for a prime p. Suppose G, a subgroup of* Aut $\mathcal{H}$, *has rank 4 on points. Then G contains as a normal subgroup a group isomorphic to $G_2(p)$ and $\mathcal{H}$ is isomorphic to the usual hexagon associated with $G_2(p)$.*

A group G having a BN-pair whose Weyl group is D_{12} naturally acts as an automorphism group of a generalized hexagon of order (s, t) with $s > 1$ and $t > 1$. Thus as an immediate consequence of Theorem 1.2 we have:

COROLLARY 1.3. *Let G be a finite group having* BN-*pair and Weyl group D_{12}. Suppose that $|P: B| - 1$ is a prime p for each maximal parabolic subgroup P. Then G has a normal subgroup H isomorphic to $G_2(p)$, with the usual* BN-*pair induced on H.*

The basic idea for the proof is to investigate the subgroup structure of a Sylow p-subgroup P of G, in particular to determine the structure of the stabilizers in P of various sets of $\mathcal{H}$. The method of attack is similar to recent work of Kantor [4] on generalized quadrangles with a prime parameter. Kantor's main result is that if Q is a generalized quadrangle of order (p, t) with p prime and if $G =$ Aut Q has rank 3 on points, then with some possible exceptions, $G \cong \mathrm{PS}p(4, p)$ or PSU$(4, p)$ and Q is one of the usual quadrangles associated with these groups. So Theorem 1.2 is a version of Kantor's theorem for generalized hexagons with equal prime parameters. We note that in order to consider the problem of a generalized hexagon with parameters (s, p) for a prime p, a geometric characterization of hexagons of type (t^3, t) is needed.

Finally the author wishes to thank William Kantor for suggesting the problem to him and for his helpful comments.

2. Lines of a generalized hexagon

Let $\mathcal{H}$ be a generalized hexagon of order (s, t) for $s, t > 1$. If z is a point or an edge of $\mathcal{H}$, define for $i = 1, 2, \cdots, 6$ the sets $\Gamma_i(z)$ by

$$\Gamma_i(z) = \{y \in \mathcal{H}: d(y, z) = i\}.$$

For a point x of $\mathcal{H}$, define the *block* containing x, denoted $x^\perp$, by

$$x^\perp = \{x\} \cup \Gamma_2(x) \cup \Gamma_4(x).$$

For two distinct points x, y of $\mathcal{H}$, define the *line* through x and y, denoted $R(xy)$, as follows:

$$R(xy) = \cap \{z^\perp : z \text{ is a point of } \mathcal{H} \text{ with } x, y \in z^\perp\}.$$

Note that $x, y \in z^\perp$ iff $z \in x^\perp \cap y^\perp$ and that $u \in R(xy)$ iff $u^\perp \supseteq x^\perp \cap y^\perp$.

Call the line $R(xy)$ *singular, hyperbolic* or *imaginary* according as $d(x, y) = 2$, 4 or 6 respectively.

If $d(x, y) = 4$, then let $x \wedge y$ denote the unique point of $\mathcal{H}$ with $d(x, x \wedge y) = 2 = d(x \wedge y, y)$.

LEMMA 2.1. *If $d(x, y) = 2$ and L is the edge of $\mathcal{H}$ joining x and y, then $R(xy) = \Gamma_1(L)$.*

PROOF. Let $u \in L$ and let $x, y \in z^\perp$. If $z \in \Gamma_2(x) \cup \{x\}$, then $d(z, u) \leqq d(z, x) + d(x, u) = 4$ and $u \in z^\perp$. Similarly if $z \in \Gamma_2(y) \cup \{y\}$, then $u \in z^\perp$. Assume now that $z \in \Gamma_4(x) \cap \Gamma_4(y)$. Then $z \wedge x = z \wedge y$; otherwise $\mathcal{H}$ contains the pentagon with vertices $x, x \wedge z, z, y \wedge z, y$. In fact $z \wedge x \in L$; otherwise $\mathcal{H}$ contains the triangle with vertices $x, x \wedge z, y$. Since $u \in L$, it follows that $d(u, z \wedge x) \leqq 2$, that $u \in z^\perp$ and that $\Gamma_1(L) \subseteq R(xy)$.

Conversely if $u \in R(xy)$, then $u^\perp \supseteq x^\perp \cap y^\perp$. Suppose that $u \in \Gamma_4(x)$. Then $x \wedge u \in L$ lest $d(u, y) = 6$. If $w \in \Gamma_2(x) \cap \Gamma_4(x \wedge u)$, then $d(w, u) = 6$ but $w \in x^\perp \cap y^\perp \subseteq u^\perp$, a contradiction. So $u \in \Gamma_2(x) \cup \{x\}$. Similarly $u \in \Gamma_2(y) \cup \{y\}$. It follows that $u \in L$; otherwise x, u, y are the vertices of a triangle of $\mathcal{H}$. Thus $R(xy) \subseteq \Gamma_1(L)$.

Note that singular lines correspond to edges of $\mathcal{H}$ and that if $d(xy) = 2$, then $R(xy) = (\Gamma_2(x) \cap \Gamma_2(y)) \cup \{x, y\}$.

Assume the number of points of a hyperbolic line is constant.

LEMMA 2.2. *Let $d(x, y) = 4$. Then*

(i) *$R(xy) \subseteq \Gamma_2(x \wedge y) \cap \Gamma_4(w)$ for any $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(x \wedge y)$. In fact the hyperbolic line $R(xy)$ consists of at most one point from each singular line through $x \wedge y$. Thus $|R(xy)| \leqq t + 1$.*

(ii) *Let $h + 1 = |R(xy)|$. Then $h \leqq t$ and h divides st.*

PROOF. (i) Let $u \in R(xy)$ and let $z = x \wedge y$. Then $z \in x^\perp \cap y^\perp \subseteq u^\perp$. If $u \in \Gamma_4(z)$, then $u \wedge z$ lies on an edge L through z. Since $d(x, y) = 4$, WLOG assume $x \notin L$. Then $d(u \wedge z, x) = 4$ and $d(u, x) = 6$, a contradiction. So $u \in \Gamma_2(z)$ and $R(xy) \subseteq \Gamma_2(z)$.

If $w \in \Gamma_4(x) \cap \Gamma_4(y) \cap \Gamma_6(z)$, then clearly $R(xy) \subseteq w^\perp$. If $v \in R(xy) \cap (\{w\} \cup \Gamma_2(w))$, then WLOG assume $v \neq x$. Since $R(xy) \subseteq \Gamma_2(z)$, it follows that $d(z, v) = 2$ and $d(z, w) \leqq 4$, a contradiction of the choice of w. So $R(xy) \subseteq \Gamma_4(w)$.

Now $\Gamma_2(z) \cap \Gamma_4(w)$ consists of exactly one point $\neq z$ from each singular line through z. Indeed, if L is a singular line on z, then $d(L, w) = 5$ because $d(z, w) = 6$. There is a unique point $b \in L - \{z\}$ with $d(b, w) = 4$ since $\mathcal{H}$ is a generalized hexagon. So $|\Gamma_2(z) \cap \Gamma_4(w)| = t + 1$ and $h \leqq t$.

(ii) $\Gamma_2(z) \cap \Gamma_4(x)$ is a union of hyperbolic lines on x with x removed and has order st. Indeed $\Gamma_2(z) \cap \Gamma_4(x)$ is the set of t singular lines on z, distinct from the singular line $R(xz)$ and so has order ts. If $w \in \Gamma_2(z) \cap \Gamma_4(x)$, then $R(xw) - \{x\} \subseteq \Gamma_2(z) \cap \Gamma_4(x)$. To see this, note that by (i) $R(xw) \subseteq \Gamma_2(z)$. If $u \in R(xw) - \{x\}$, then $u \in \Gamma_4(x)$. Otherwise if $u \in \Gamma_2(x)$, then $u \in R(xz) - \{x, z\}$, which contradicts (i). But $\Gamma_6(x) \cap \Gamma_2(z) = \emptyset$ if $d(x, z) = 2$. So $u \in \Gamma_4(x)$.

Now express $\Gamma_2(z) \cap \Gamma_4(x)$ as a disjoint union of sets $R(xw) - \{x\}$ for $w \in \Gamma_2(z) \cap \Gamma_4(x)$ to see that h divides st. This is possible due to the following Lemma 2.3.

LEMMA 2.3. *If $d(x, y) = 4$ and if $u \in R(xy) - \{x\}$, then $R(xu) = R(xy)$.*

PROOF. If $u \in R(xy) - \{x\}$, then $x^\perp \cap u^\perp \supseteq x^\perp \cap y^\perp$. By Lemma 2.2 (i) $d(x, u) = 4$ and so $|x^\perp \cap u^\perp| = |x^\perp \cap y^\perp|$. So $x^\perp \cap u^\perp = x^\perp \cap y^\perp$ and $R(xu) = R(xy)$.

From the last two lemmas, it follows that two points at distance 4 are on exactly one hperbolic line and on no singular line and that two distinct hyperbolic lines meet at most in one point. Clearly a similar statement holds for singular lines.

LEMMA 2.4. (i) *If $d(x, y) = 2$, then for $\mathcal{P}$ the set of points of $\mathcal{H}$*

$$\mathcal{P} = \cup \{z^\perp : z \in R(xy)\}.$$

(ii) *If $d(x, y) = 4$, then $|R(xy)| \leqq s + 2$. Also $\mathcal{P} = \cup \{z^\perp : z \in R(xy)\}$ iff $s = t = h$.*

PROOF. (i) If $a \in \mathcal{P}$, then $\delta = d(a, R(xy)) \in \{1, 3, 5\}$ and there is a unique $z \in R(xy)$ with $d(a, z) = \delta - 1$. Thus $a \in z^\perp$.

(ii) Let $R(xy) = \{a_0, a_1, \cdots, a_h\}$. Let $A = \bigcup_{i=0}^{h} a_i^\perp$. Now express A as a pairwise disjoint union of sets as follows:

$$A = a_0^\perp \cup \bigcup_{i=1}^{h} (a_i^\perp \cap \Gamma_6(a_0)).$$

Then $|A| = (1+(t+1)s + s^2(t+1)t) + hst(st+s-t)$, since $|a_i^\perp \cap \Gamma_6(a_0)| = |a_i^\perp| - |a_i^\perp \cap a_0^\perp|$. Now

$$|\mathcal{P}| - |A| = st(s^2t - h(st+s-t)) \geqq 0,$$

which implies the desired results.

LEMMA 2.5. *If $s = t = h$, then $\{x\} \cup \Gamma_2(x)$ together with the singular and hyperbolic lines contained in this set is a projective plane of order t.*

PROOF. $\{x\} \cup \Gamma_2(x)$ has $1+(1+t)t$ points, $1+t$ singular lines and t^2 hyperbolic lines. Indeed count the pairs (y, H) with $y \in H \subseteq \Gamma_2(x)$ for H a hyperbolic line. From the proof of Lemma 2.2 (ii) each y lies on $st/h = t$ hyperbolic lines. Thus $\{x\} \cup \Gamma_2(x)$ contains $1+t+t^2$ lines, each of which contains $1+t$ points. Furthermore any 2 points determine a unique line. Therefore $\{x\} \cup \Gamma_2(x)$ is a projective plane with the property that if $y, z \in \{x\} \cup \Gamma_2(x)$, then $d(y,z) \leqq 4$ and $y \in z^\perp$.

Note that the previous lemma is needed only for the proof of Theorem 1.2.

3. The proof of Theorem 1.1

Assume $\mathcal{H}$ is a generalized hexagon of order (t,t). Assume the hyperbolic lines of $\mathcal{H}$ carry $t+1$ points. Let $\mathcal{P}$ be the set of points of $\mathcal{H}$. Let $\mathcal{L}$ be the set of singular and hyperbolic lines formed from $\mathcal{P}$. We will show that $(\mathcal{P}, \mathcal{L})$ is a non-degenerate pre-polar space. Since $\mathcal{H}$ is a generalized hexagon, clearly $(\mathcal{P}, \mathcal{L})$ is non-degenerate.

Let p, L be a non-incident point-line pair. Since by Lemma 2.4,

$$\mathcal{P} = \cup \{z^\perp : z \in L\},$$

there is $z \in L$ with $p \in z^\perp$. If there is $z' \in L - \{z\}$ with $p \in z'^\perp$, then $z, z' \in p^\perp$ and $L = R(zz') \subseteq p^\perp$ by Lemma 2.3. So p is incident either to one point or to all points of L.

Since $(\mathcal{P}, \mathcal{L})$ is a finite incidence structure, $(\mathcal{P}, \mathcal{L})$ is a pre-polar space of finite rank in which lines carry $t+1$ points and which has $(t^6-1)/(t-1)$ points. By Theorem 4 of [1], $\mathcal{P}$ together with its subspaces is a polar space of finite rank. If $\mathcal{P}$ has rank 2, then $\mathcal{P}$ is a generalized quadrangle of order (t, t^3+t^2+t) because lines of $(\mathcal{P}, \mathcal{L})$ carry $t+1$ points, lines are maximal subspaces and each point lies in $1+t$ singular and $t^2(t+1)$ hyperbolic lines. But a quadrangle of order (t, t^3+t^2+t) contains $(1+t)(1+t(t+t^2+t^3))$ points while $|\mathcal{P}| =$

$(1+t)(1+t^2+t^4)$, a contradiction. So $\mathcal{P}$ has rank $\geqq 3$. By theorem 1 of [1], $\mathcal{P}$ is a polar space associated with a classical geometry of symplectic, unitary or orthogonal type. In particular t is a prime power. Since $|\mathcal{P}| = (t^6 - 1)/(t-1)$, it follows that $\mathcal{P}$ is either a symplectic geometry of dimension 6 or an orthogonal geometry of dimension 7.

If t is even, then $\mathcal{P}$ is an orthogonal geometry of dimension 7 which is isomorphic to a symplectic geometry of dimension 6. If t is odd, then there are two cases to consider. First assume that no imaginary line contains more than two points. Then $\mathcal{P}$ is an orthogonal geometry of dimension 7. Secondly assume that some imaginary line contains more than 2 points. Then $\mathcal{P}$ is a symplectic geometry and so every imaginary line contains more than 2 points. Now there is a plane of the generalized hexagon $\mathcal{H}$ which contains a quadrangle whose diagonal points are collinear. The proof of this statement is found as (13.5) in Schellekens [5]. Thus the characteristic of the underlying field of the polar space is two. This contradicts the assumption that t is odd. Therefore $\mathcal{P}$ is an orthogonal geometry of dimension 7.

The final step of the proof is to identify the generalized hexagon $\mathcal{H}$, which is now embedded in the orthogonal geometry, with the generalized hexagon associated with $G_2(t)$. For this we refer the reader to section 14 of Schellekens [5].

4. Rank 4 subgroups of Aut $\mathcal{H}$

Let $\mathcal{H}$ be a generalized hexagon of order (s,t). H_S will denote the set of elements of $H \leqq \mathrm{Aut}\,\mathcal{H}$ fixing the set S, where S is a set of points or lines of $\mathcal{H}$. $H(S)$ will denote the set of elements of H fixing each element of S.

LEMMA 4.1. *Suppose $G \leqq \mathrm{Aut}\,\mathcal{H}$ has rank 4 on points. Then G_L is 2 transitive on the points of L, a line of $\mathcal{H}$.*

PROOF. This is an immediate consequence of the facts that G_x is transitive on $\Gamma_2(x)$ and that $s > 1$.

LEMMA 4.2. *Let $G = \mathrm{Aut}\,\mathcal{H}$ have rank 4 on the points of $\mathcal{H}$.*

(i) *If $d(x,u) = 2$, then G_{xu} is transitive on $\Gamma_4(x) \cap \Gamma_2(u)$.*

(ii) *If L is the singular line through x and u, then G_L is transitive on the points of $\Gamma_3(L)$.*

(iii) *If $d(x,u) = 2$ and if $(s, t+1) = 1$, then G_{xu} is transitive on $x^{\perp} \cap \Gamma_6(u) = \Gamma_4(x) \cap \Gamma_6(u)$.*

(iv) *If $(s, t+1) = 1$, then G_L is transitive on $\Gamma_5(L)$.*

PROOF. (i) If $v \in \Gamma_4(x) \cap \Gamma_2(u)$, then $G_{xv} = G_{xvy}$ since $y = x \wedge v$. Since G is rank 4 on points, $|G_x : G_{xv}| = |\Gamma_4(x)| = s^2t(t+1)$ and $|G_x : G_{xy}| = s(t+1)$. Thus $|G_{xy} : G_{xyv}| = st = |\Gamma_4(x) \cap \Gamma_2(y)|$.

(ii) Let $z \in \Gamma_3(L)$. Let y be the unique point of L with $d(y, z) = 2$. Say $y \neq x$. By part (i), $|z^{G_{xy}}| = st$. Since G_L is transitive on the points of L by Lemma 4.1, $|G_L : G_{Lx}| = s + 1$. Since $G_{Lx} \geqq G_{xy}$, it follows that $|z^{G_L}| = (s+1)st = |\Gamma_3(L)|$.

(iii) Note that $|x^\perp \cap \Gamma_6(u)| = s^2t^2$. For if $z \in x^\perp \cap \Gamma_6(u)$, then $d(z, L) = 5$ and x is the unique point of L with $d(z, x) = 4$. Since G is rank 4 on points, $|G_u : G_{uz}| = s^3t^2 = |\Gamma_6(u)|$ and $|G_u : G_{ux}| = s(t+1) = |\Gamma_2(u)|$. Then

$$|G_u : G_{uxz}| = s(t+1) \cdot |G_{ux} : G_{uxz}| = s^3t^2 \cdot |G_{uz} : G_{uxz}|.$$

Since $(s, t+1) = 1$ by hypothesis, s^2t^2 divides $|G_{ux} : G_{uxz}|$. But

$$z^{G_{xu}} \subseteq x^\perp \cap \Gamma_6(u),$$

a set of order s^2t^2. Thus $s^2t^2 = |G_{ux} : G_{uxz}|$.

(iv) The proof follows from (iii) just as (ii) follows from (i).

LEMMA 4.3. *Suppose $G \leqq \operatorname{Aut} \mathcal{H}$ has rank 4 on points. Then G_x is doubly transitive on the lines through x.*

PROOF. If x and u are distinct points of a line L, then G_{xL} contains G_{xu}, which is transitive on the points of $\Gamma_4(u) \cap \Gamma_2(x)$ by Lemma 4.2 (i). This implies the desired result.

LEMMA 4.4. *Suppose $G \leqq \operatorname{Aut} \mathcal{H}$ has rank 4 on points. Let $d(x, u) = 2$. Then* (i) *the pointwise stabilizer $G(x^\perp \cap u^\perp)$ is semiregular off $x^\perp \cap u^\perp$.*

If in addition $|G(x^\perp \cap u^\perp)| = t$, then

(ii) *for $w \in \Gamma_4(u) \cap \Gamma_6(x)$, the group $G(x^\perp \cap u^\perp)$ is regular on $\Gamma_1(w) - \{R(u, u \wedge w)\}$.*

(iii) *Hyperbolic lines carry $t + 1$ points.*

(iv) *The group $G(x^\perp \cap u^\perp)$ is regular on $R(vw) - \{v\}$.*

PROOF. (i) See (7.4) of [5].

(ii) Let H be the hexagon with vertices x, u, v, w, b, a. A nontrivial element $g \in G(x^\perp \cap u^\perp)$ does not fix $R(vw)$ setwise. Otherwise there is a 5-gon with vertices $w, g(w), g(b), a, b$, which is impossible. So g moves $R(vw)$ to another singular line on v. If the orbit of $R(vw)$ under $G(x^\perp \cap u^\perp)$ is a proper subset of $\Gamma_1(v) - \{R(vu)\}$, then $G(x^\perp \cap u^\perp)_{R(vw)} \neq 1$ since

$$|G(x^\perp \cap u^\perp) : G(x^\perp \cap u^\perp)_{R(vw)}| = |R(vw)^{G(x^\perp \cap u^\perp)}|.$$

But a nontrivial element of $G(x^\perp \cap u^\perp)_{R(vw)}$ again implies the existence of a 5-gon. So $|G(x^\perp \cap u^\perp)|$ divides $|\Gamma_1(v) - \{R(vu)\}|$ and the additional assumption that $t = |G(x^\perp \cap u^\perp)|$ gives the desired regular action.

(iii) Let H be the hexagon with vertices x, u, v, w, b, a. Clearly the hyperbolic line $R(uw)$ is a subset of $\Gamma_2(v) \cap \Gamma_4(a)$. The claim is that the two sets are equal. For $z \in \Gamma_2(v) \cap \Gamma_4(a)$, it follows that $z \in R(uw)$ iff $z^\perp \supseteq u^\perp \cap w^\perp$. It suffices to show that $z \in i^\perp$ for $i \in u^\perp \cap w^\perp$ in order to establish the claim.

If $d(v, i) = 2$, then by the triangle inequality $d(z, i) \leqq 4$. Note that $d(v, i) \neq 4$. Otherwise the unique path joining v and i does not contain one of $R(vu)$ and $R(vw)$, say $R(vw)$. Then

$$d(i, w) = d(i, v) + d(v, w) = 6,$$

a contradiction. Now assume $d(v, i) = 6$.

Let v, u, x, a, c, z be the vertices of a hexagon H. By (ii) there is $g_1 \in G(v^\perp \cap w^\perp)$ with $g_1(R(ux)) = R(uh)$ and $g_2 \in G(v^\perp \cap u^\perp)$ with $g_2(R(wb)) = R(wj)$. Put $g = g_2 g_1$. Then g fixes u, v, w and z. If $g(a) = i$, then $d(z, i) = 4$ since $d(z, a) = 4$, and so $z \in i^\perp$. Now assume $g(a) \neq i$. Note that $d(g(a), i) = 6$ since otherwise there exists a 4- or a 5-gon. Also $d(v, g(a)) = 6$ since $d(v, a) = 6$. Note that $g(H)$ is a hexagon containing the edges $R(vu)$, $R(uh)$, $R(vw)$ and $R(wj)$. So

$$d(R(uh), R(wj) = 6 = d(g(R(ux), g(R(zc))) = d(g(R(wb)), g(R(zc))).$$

In addition

$$\begin{aligned} &d(v, R(uh)) = 3 = d(v, R(wj)) = d(v, g(R(zc)); \\ &d(g(a), R(uh)) = 3 = d(g(a), R(wj)) = d(g(a), g(R(zc))); \\ &d(i, R(uh)) = 3 = d(i, R(wj)). \end{aligned}$$

Finally the claim is that $d(i, g(R(zc)) = 3$ and so $d(i, z) = 4$, as desired.

Indeed there is $g_3 \in G(u^\perp \cap h^\perp)$ with $g_3(R(vw)) = R(vz)$ by (ii). Since $g_3(g(a)) = g(a)$, it follows that $g(R(wj)) = R(zg(c))$. Thus

$$d(i, R(zg(c))) = d(g_3(i), g_3(R(wj)) = d(i, R(wj)) = 3.$$

(iv) $G(x^\perp \cap u^\perp)$ acts semiregularly on $\Gamma_2(v) \cap \Gamma_4(a) - \{u\}$ by (i) and this set equals $R(uw) - \{u\}$ by (iii). By computation of orders, there is regular action.

LEMMA 4.5. *Assume $G \leqq \mathrm{Aut}\,\mathcal{H}$ has rank 4 on points. Assume that $s = t = |G(x^\perp \cap u^\perp)|$ where $d(x, u) = 2$. Then $\mathcal{H}$ is isomorphic to the usual hexagon for $G_2(s)$ and $G \trianglerighteq G_2(s)$.*

PROOF. By Theorem 1.1, $\mathcal{H}$ is isomorphic to the usual hexagon associated with $G_2(s)$ because by hypothesis $s = t$ and by Lemma 4.4 (iv) $t = h$. Under this isomorphism the set $x^\perp$ for $x \in \mathcal{P}$ corresponds to the set of singular points of the orthogonal geometry of dimension 7. Furthermore the elements of $G(x^\perp \cap u^\perp)$ correspond to Siegal transformations of root type 1. (See [6] for the definition and properties of these transformations.) The pair consisting of the point set $\{x\} \cup \Gamma_2(x)$ and of the lines in this set corresponds to a projective plane of order s, which is a singular subspace of the orthogonal geometry by Lemma 2.5. Thus G contains elements which correspond to all the Siegal transformations that are in $G_2(s)$. Since these transformations generate $G_2(s)$,

$$\langle G(x^\perp \cap u^\perp): x, u \in \mathcal{P} \quad \text{with} \quad d(x, u) = 2 > \cong G_2(s),$$

as desired. Note that a theorem of Stark [6] also implies the desired result.

LEMMA 4.6. *If $g \in \mathrm{Aut}\,\mathcal{H}$ fixes all points on a line L, all lines through a point $x \in L$ and also fixes some line K with $d(L, K) = 6$, then $g = 1$.*

PROOF. This is a result due to J. Tits. For a proof see Lemma 1 of Faulkner [2].

5. The proof of Theorem 1.2

Let $\mathcal{H}$ be a generalized hexagon of order (p, p) where p is a prime. Suppose $G = \mathrm{Aut}\,\mathcal{H}$ has rank 4 on the vertices of $\mathcal{H}$. If $p = 2$, then $\mathcal{H}$ is isomorphic to the usual hexagon for $G_2(2)$ by Theorem 11.5 of Tits [7]. Assume $p > 2$. Let P be a Sylow p-subgroup of G. Then $|P| \geqq p^5$ since G_x is transitive on $\Gamma_6(x)$, which has order p^5. The group P fixes some point x because the number of points of $\mathcal{H}$ is $(1 + p)(1 + p^2 + p^4)$. Also P fixes some singular line L on x since the number of singular lines on x is $1 + p$. Let $L = R(xu)$. Choose points v, w, b and a of $\mathcal{H}$ so that x, u, v, w, b, a are the vertices of a hexagon of $\mathcal{H}$, which is denoted H. Let $N = R(xa)$ and $M = R(uv)$.

The goal of the proof is to show that $P(x^\perp \cap u^\perp) \neq 1$ for vertices x, u with $d(x, u) = 2$. If $P(x^\perp \cap u^\perp) \neq 1$, then by Lemma 4.4 (ii) and (iii), $|P(x^\perp \cap u^\perp)| = h = p$ and by Lemma 4.5, $G \trianglerighteq G_2(p)$ and $\mathcal{H}$ is the hexagon for $G_2(p)$.

LEMMA 5.1. *If g is a p-element of G which fixes each vertex of a hexagon of $\mathcal{H}$, then $g = 1$.*

PROOF. Suppose g fixes the hexagon with vertices $v_1, v_2, \cdots, v_6$. Then $g \in G(\Gamma_1(R(v_1v_2)) \cap G(\Gamma_1(v_1))$. Furthermore g fixes $R(v_4v_5)$ with $d(R(v_1v_2), R(v_4v_5)) = 6$. By Lemma 4.6 it follows that $g = 1$.

LEMMA 5.2. *The group P is transitive on $L-\{x\}$, on $\Gamma_6(x)$, on $\Gamma_2(x)-L$ and has two orbits on $\Gamma_4(x)$, one of length p^3 and one of length p^4. Furthermore $|P_a:P_{au}|=p$ and $|P|=p^5$ or p^6.*

PROOF. (i) P is transitive on $L-\{x\}$. For $|G_{Lx}:G_{xu}|=p$ since G_L is doubly transitive on the points of L by Lemma 4.1. Then

$$|G_{Lx}:P_u|=p\cdot|G_{xu}:P_u|=|G_{Lx}:P|\cdot|P:P_u|,$$

where $|G_{Lx}:P|$ is a p'-number because P is Sylow in G_{xL}. Thus $p=|P:P_u|$.

(ii) P is transitive on $\Gamma_6(x)$. The proof is similar to (i) and omitted.

(iii) P is transitive on $\Gamma_2(x)-L$, a set of order p^2. Indeed $|G_x:G_{xa}|=(p+1)p$ because G_x is transitive on $\Gamma_2(x)$. Then

$$|G_x:P_a|=(p+1)p\cdot|G_{xa}:P_a|=|G_x:P|\cdot|P:P_a|.$$

So p divides $|P:P_a|$. Suppose $p=|P:P_a|$. Note that

$$|P_a:P_{aw}|\leqq|\Gamma_4(a)\cap\Gamma_6(x)|=p^4.$$

It follows that $|P_a:P_{aw}|=p^4$ and that $P_w=P_{wa}$ because $|P:P_w|=p^5$ by (ii). So P_w fixes $a\wedge w=b$, $L\cap w^\perp=u$, $u\wedge w=v$ and each vertex of the hexagon H. Thus $P_w=1$ by Lemma 5.1, and by (ii) $|P|=p^5$. Now no non-trivial p-element can fix 2 points at maximal distance.

But $|P:P_a|=p$ and $|P_{ab}|\geqq p^2$ because

$$|P_a:P_{ab}|\leqq|\Gamma_2(a)\cap\Gamma_4(x)|=p^2.$$

Since P_{ab} fixes L setwise, $|P_{abu}|\geqq p$ where $d(b,u)=6$, a contradiction. So $|P:P_a|=p^2$ and (iii) holds.

(iv) $|P:P_v|=p^3$. Indeed $|G_x:G_{xv}|=(p+1)p^3$ since G_x is transitive on $\Gamma_4(x)$. Then

$$|G_x:P_v|=(p+1)p^3\cdot|G_{xv}:P_v|=|G_x:P|\cdot|P:P_v|,$$

where $|v^P|$ is a p-power less than or equal to $|\Gamma_4(x)\cap\Gamma_3(L)|=p^3$. Thus $p^3=|P:P_v|$.

(v) $|P:P_b|=p^4$. Indeed

$$|G_x:P_b|=(p+1)p^3\cdot|G_{xb}:P_b|=|G_x:P|\cdot|P:P_b|,$$

where $|b^P|$ is a p-power less than or equal to $|\Gamma_4(x)\cap\Gamma_5(L)|=p^4$. If $|P:P_b|=p^3$, then $|P_b|\geqq p^2$ and $|P_{bu}|\geqq p$ where $d(b,u)=6$. So $P_w\neq 1$ for $P_w=1$ implies that no non-trivial p-element fixes 2 points at maximal distance. Thus $|P|\geqq p^6$, $|P_{bu}|\geqq p^2$ and $|P_{buR(uv)}|\geqq p$. Now $P_{buR(uv)}$ fixes $R(uv)\cap b^\perp=v$, then $v\wedge b=w$

and each vertex of the hexagon H. By Lemma 5.1, $P_{buR(uv)} = 1$, a contradiction. Thus $|P: P_b| = p^4$.

(vi) Note that $|P_a: P_{au}| = p$. Otherwise $P_a = P_{au} \leqq P_u \lhd P$ and so P_u fixes a^P. Consequently $P_u = P(\Gamma_2(x))$, which contradicts (iii). So $|P_a: P_{au}| = p$ and then $|P_u: P_{au}| = p^2$.

(vii) By (ii) $|P: P_w| = p^5$. If $P_w = P_{w,R(wb)}$, then P_w fixes $x^\perp \cap R(wb) = b$, then fixes $b \wedge x = a$, so fixes each vertex of H and by Lemma 5.1 is trivial. So $|P_w| = 1$ or p.

LEMMA 5.3. *If $Z(P)$ denotes the center of P, then $Z(P) \cap P(\Gamma_1(x)) \cap P(\Gamma_1(L)) \neq 1$.*

PROOF. Note first that $P(\Gamma_1(x)) = P_N$ and $P(\Gamma_1(L)) = P_u$. Now $|P: P_N| = p$ because G_x is 2-transitive on the lines through x by Lemma 4.3. So $|P_N| \geqq p^4$ since $|P| \geqq p^5$. Similarly $|P_u| \geqq p^4$. If $P_N = P_u$, then $P_{ub} \neq 1$ because $|P_N: P_{ab}| = p^3$ by Lemma 5.2. So $P_w \neq 1$ and $|P| = p^6$. But $|P_{NR(vw)}| \geqq p^2$ since $|P_u: P_{u,R(vw)}| = p^3$. Hence $|P_{Nw}| \geqq p$ and P_{Nw} fixes $N \cap w^\perp = a$, then $a \wedge w = b$, so fixes each vertex of the hexagon H. By Lemma 5.1, $P_{Nw} = 1$, a contradiction. Therefore $P_N \neq P_u$ and $P = P_N P_u$. Since $|P_{Nu}| \geqq p^3$ and P_{Nu} is normal in P as P_{Nu} is the intersection of two normal subgroups of P, it follows that $Z(P) \cap P_{Nu} \neq 1$.

LEMMA 5.4 (i) *If $P_w \neq 1$, then P_w is regular on the set of singular lines through w, unequal to $R(wv)$.*

(ii) *If $P_w = 1$, then P is transitive on $\Gamma_6(L)$.*

PROOF. (i) From (vii) of the proof of Lemma 5.2, $P_{wR(wb)} = 1$. Since $|P_w| = p$, the result follows.

(ii) If $P_w = 1$, then $|P| = p^5$. If $K \in \Gamma_6(L)$, then there are $p+1$ paths of length 6 from L to K. Let γ denote the path which goes through $u \in L$. Let v and w denote the points of γ with $d(L, v) = 3$ and $d(L, w) = 5$, respectively. By Lemma 5.2, $|P: P_v| = p^3$. So $|P_v: P_{R(vw)}| = p$. For if $P_v = P_{R(vw)}$, then $|P_w| \geqq p$. Thus $|P: P_{R(vw)}| = p^4$. Since $P_{R(vw),K} \leqq P_w = 1$, it follows that $|P_{R(vw)}: P_{R(vw),K}| = p$ and that $|P: P_{R(vw),K}| = p^5 = |\Gamma_6(L)|$.

LEMMA 5.5. *The group G has rank 4 on the singular lines of $\mathscr{H}$.*

PROOF. The group G is transitive on the singular lines of $\mathscr{H}$. Indeed if $R(ab)$ and $R(cd)$ are singular, then there is $g \in G$ with $g(a) = c$. Since $g(b), d \in \Gamma_2(c)$, an orbit of G_c, there is $h \in G_c$ with $h(g(b)) = d$. So $hg(R(ab)) = R(cd)$.

Now fix a singular line L. If M_1 and M_2 are singular lines of $\Gamma_2(L)$, then for $u_i \in M_i$ with $d(u_i, L) = 3$, there is $g \in G_L$ with $g(u_1) = u_2$ because G_L is transitive on the points of $\Gamma_3(L)$ by Lemma 4.2. Since $\mathcal{H}$ is a generalized hexagon, the unique path from L to u of length 3 must be sent by $g \in G_L$ to the unique path of length 3 from L to u_2. It follows that $g(M_1) = M_2$ and that $\Gamma_2(L)$ is an orbit of G_L.

Since G_L is transitive on $\Gamma_5(L)$ by Lemma 4.2, it follows that G_L is transitive on $\Gamma_4(L)$.

Let $M_1, M_2 \in \Gamma_6(L)$. There are $1 + p$ paths of length 6 from L to M_i. Fix paths γ_i of length 6 from L to M_i which pass through $u \in L$. If $w_i \in \gamma_i$ with $d(L, w_i) = 5$, then there is $g \in G_L$ with $g(w_1) = w_2$ and $g(\gamma_1^*) = \gamma_2^*$, where γ_i^* denotes the subpath of γ_i from L to w_i. If $g(M_1) = M_2$, we are done. Assume now that $g(M_1) \neq M_2$. If $P_w \neq 1$, then by Lemma 5.4 (i), there is $h \in P_w$ with $h(g(M_1)) = M_2$. If $P_w = 1$, then by Lemma 5.4 (ii), P is transitive on $\Gamma_6(L)$. In either case, $\Gamma_6(L)$ is an orbit of G_L. Therefore G has rank 4 on the lines of $\mathcal{H}$, as desired.

LEMMA 5.6. *Let $P_v = P(\Gamma_2(u))$. Let i, j, k be points of $\mathcal{H}$ with $d(i, k) = 4$ and $i \wedge k = j$. Let S by a Sylow p-subgroup of $G_{j,R(jk)}$. Then $S_{ik} = S(\Gamma_2(j))$.*

PROOF. First we show that $P_{ua} = P(\Gamma_2(x))$. Since G is rank 4 on points, G is transitive on ordered pairs of hyperbolic points at distance 4. There is $g \in G$ with $g(x) = a$ and $g(v) = u$. Note that

$$x = a \wedge u = g(x) \wedge g(v) = g(x \wedge v) = g(u).$$

Since $P_v = P(\Gamma_2(u))$, it follows that $(P_v)^g = P^g \cap G(\Gamma_2(x))$, a subgroup which fixes x and L. By Sylow's Theorem, there is $h \in G_{xL}$ with $(P^g)^h = P$. By Lemma 5.2, there is $k_1 \in P_{h(u)}$ with $k_1(h(a)) = a$ because

$$| P_u : P_{ua} | = | P_{h(u)} : P_{h(u),h(a)} | = p^2.$$

Also by Lemma 5.2, there is $k_2 \in P_a$ with $k_2(h(u)) = u$ since $| P_a : P_{au} | = p$. Now let $l = k_2 k_1 hg$. Then

$$P_v^l = P \cap G(\Gamma_2(x)) = P_{xv}^l = P_{ua},$$

as desired. By a similar argument, it follows that

$$S_{ik} = S(\Gamma_2(j)).$$

LEMMA 5.7. *Let $P_b = P(\Gamma_2(a))$. Then either $S_{ik} = S(\Gamma_2(j))$ or $S_{ikR(kk')} = S(\Gamma_2(j))$ where $d(k, k') = 2$ and $d(k, j) = 4$.*

PROOF. Note $P_b = P_{xb} \cap P_{R(xu)} = P(\Gamma_2(a)) \cap P_{R(xu)}$. By Lemma 5.6, $P_b^l = P_{ua} \cap P_{R(uu')} = P(\Gamma_2(x)) \cap P_{R(uu')}$ where $d(u, u') = 2$ and $d(x, u') = 4$. Since

$|P_{ua}:P_{uaR(uv)}|=p$, there is $k_3\in P_{ua}$ with $k_3(R(uu'))=R(uv)$ and so $P_{uaR(uv)}=P(\Gamma_2(x))_{R(uv)}$. Since $P_{ua}\geqq P(\Gamma_2(x))$ and $|P_{ua}:P_{uaR(uv)}|=p$, it follows that either $P_{ua}=P(\Gamma_2(x))$ or $P_{uaR(uv)}=P(\Gamma_2(x))$. Note that in the latter case, $P(\Gamma_2(x))$ fixes $R(uv)^P$ and so fixes $\Gamma_2(L)$ elementwise. Thus $P(\Gamma_2(x))\leqq P(\Gamma_2(L))$. The desired result follows by a similar argument.

LEMMA 5.8. *Either* $P(\Gamma_2(x))=P_{ua}$ *or* $P(\Gamma_2(x))=P_{uaR(uv)}$. *In particular* $P(\Gamma_2(x))\neq 1$.

PROOF. If Z denotes $Z(P)\cap P(\Gamma_1(x))\cap P(\Gamma_1(L))$, then $Z\neq 1$ by Lemma 5.3. Assume first that $Z>Z_a$. Then Z is transitive on $R(xa)-\{x\}$. The group $P_{R(ab)}$ lies in $P(\Gamma_1(x))\cap P(\Gamma_1(a))$ and is normalized by Z So $P_{R(ab)}\leqq P(\Gamma_1(a'))$ for all $a'\in R(xa)$ and $P_{R(ab)}=P(\Gamma_2(R(xa)))$ in $\mathcal{H}$. Now G is rank 4 on the lines of $\mathcal{H}$. By the dual of Lemma 5.6, $P(\Gamma_2(L))=P_{R(uv),R(xa)}$ since $R(uv)\wedge R(xa)=L$. The result follows.

Now assume that $Z=Z_a\neq 1$. If in addition Z_a lies in $P(\Gamma_1(a))\cap P(\Gamma_1(u))$, then Z_a fixes $R(ab)^P$ and $R(uv)^P$ and so fixes each element of $\Gamma_3(x)=R(xa)^\perp\cap R(xu)^\perp$. Therefore $Z_a=P(\Gamma_3(x))\neq 1$. Now apply Lemma 4.5 to $\mathcal{H}^*$, the dual of $\mathcal{H}$. Assume next that $Z_a\not\leqq P(\Gamma_1(a))\cap P(\Gamma_1(u))$. If $Z_a\not\leqq P(\Gamma_1(u))$, then P_v fixes $R(uv)^Z$ pointwise and so $P_v=P(\Gamma_2(u))$. By Lemma 5.6 $P_{ua}=P(\Gamma_2(x))$. If $Z_a\leqq P(\Gamma_1(u))$, then $Z_a\not\leqq P(\Gamma_1(a))$. The group P_b fixes $R(ab)^Z$ pointwise and so $P_b=P(\Gamma_2(a))$. By Lemma 5.7, either $P_{ua}=P(\Gamma_2(x))$ or $P_{uaM}=P(\Gamma_2(x))$.

LEMMA 5.9. *If* $P_w\neq 1$, *then* $P(x^\perp\cap u^\perp)\neq 1$.

PROOF. Let $X=Z(P)\cap P(\Gamma_2(x))$. By Lemma 5.8, it follows that $X\neq 1$. If $X_v\neq 1$, then X_v fixes v^P pointwise since X_v is a central subgroup. By Lemma 5.2, it follows that X_v fixes $\cup\{\Gamma_2(y)\cup\{y\}\colon y\in L\}=x^\perp\cap u^\perp$ pointwise. Thus $X_v=P(x^\perp\cap u^\perp)$ and now apply Lemma 4.5 to $\mathcal{H}$.

Assume now that $X_v=1$. If $X_{R(uv)}\neq 1$, then $X_{R(uv)}$ fixes $R(uv)^P$ pointwise and so $X_{R(uv)}\leqq P(\Gamma_2(L))$. Let $Z=Z(P)\cap P(\Gamma_2(x))\cap P(\Gamma_2(L))$. Then $Z=X_{R(uv)}\neq 1$. If $Z_b\neq 1$, then Z_b fixes $b^\perp\cap R(uv)=\{v\}$; but $X_v=1$, a contradiction. If $Z_{R(ab)}\neq 1$, then $Z_{R(ab)}$ fixes $R(ab)^P$. Since $|P:P_b|=p^4$ by Lemma 5.2, $Z_{R(ab)}$ fixes elementwise

$$\{x\}\cup\Gamma_3(x)=R(xu)^\perp\cap R(xa)^\perp.$$

Because G is rank 4 on singular lines by Lemma 5.5, apply Lemma 4.5 to $\mathcal{H}^*$, the dual of $\mathcal{H}$.

Assume now that $Z_{R(ab)}=1$. Then Z is regular on the set of singular lines, unequal to $R(xa)$, through a. Since P_b fixes $R(ab)$ pointwise, P_b fixes

$$R(ab)^Z \cup R(ax) = \Gamma_2(a)$$

pointwise and $P_b = P(\Gamma_2(a))$. It follows by Lemma 5.7 that either $P(\Gamma_2(x)) = P_{ua}$ or that $P(\Gamma_2(x)) = P_{uaR(uv)}$. So $P_w = P_{uwR(ux)}$ fixes $\Gamma_2(v)$ pointwise. Indeed pick S to be a Sylow p-subgroup of $G_{vR(vu)}$ with $S \geqq P_w$. By Lemma 5.6, $S_{uwR(ux)}$ fixes $\Gamma_2(v)$ pointwise and $S_{uwR(ux)} \geqq P_w$. Because $Z = X_{R(uv)}$ is regular on $R(uv) - \{u\}$, it follows that P_w fixes $\Gamma_2(v)^Z$ pointwise. Since $P_w \leqq P_{R(vw)} = P_{vxR(vw)}$ and $P_{vxR(vw)}$ fixes $\Gamma_2(u)$ pointwise, P_w fixes elementwise

$$\cup \{\Gamma_2(v') \colon v' \in R(uv)\} = u^\perp \cap v^\perp.$$

If $P_w \neq 1$, then apply Lemma 4.5 to $\mathcal{H}$.

Finally assume $X_{R(uv)} = 1$. Then X is regular on the set of singular lines through u, unequal to $R(ux)$. So P_v fixes $R(uv)^X$ pointwise and fixes $\Gamma_2(u)$ pointwise. If $v' \in \Gamma_2(u) - \{R(uv) \cup L\}$, then there is $g \in X$ with $g(R(uv)) = R(uv')$ where $d(g(v), w) = 6$. By Lemma 5.6, P_w fixes $\Gamma_2(v)$ pointwise. So $P_w^g = P_w$ fixes $\Gamma_2(g(v))$ pointwise. Now choose points y and z so that $g(v)$, u, v, w, y and z are the vertices of a hexagon of $\mathcal{H}$. Since $z \in \Gamma_2(g(v))$, it follows that P_w fixes z, so $z \wedge w = y$ and P_w fixes each vertex of the hexagon. By Lemma 5.1, $P_w = 1$. This completes the proof of the lemma.

Now assume that $P_w = 1$. So $|P| = p^5$ and no non-trivial p-element can fix 2 points at maximal distance. Since G has rank 4 on lines, $P_K = 1$ for $K \in \Gamma_6(L)$ and no non-trivial p-element can fix 2 singular lines at maximal distance.

By Lemma 5.2, $|P : P_u| = p$ and $|P : P_a| = p^2$. Now $|P : P_{ua}| = p^3$ and $|P_{ua}| = p^2$. For if $P_{ua} = P_a$, then since $|P_a : P_{ab}| = p^2$, the group P_{ab} is nontrivial and fixes b, u with $d(b, u) = 6$, a contradiction. By the principle of duality, $|P_{MN}| = p^2$. We will derive a contradiction by studying the subgroups of P_{ua} and P_{MN}. The argument is similar to Kantor's [4].

Lemma 5.10. *$P_{ua} = P(\Gamma_2(x))$ and $P_{MN} = P(\Gamma_2(L))$.*

Proof. By Lemma 5.8, either $P_{ua} = P(\Gamma_2(x))$ or $P_{uaM} = P(\Gamma_2(x))$. Suppose $P(\Gamma_2(x)) = P_{uaM}$ of order p. Let S be a Sylow p-subgroup of G_{xN} with $S \geqq P_{uaM}$. By Lemma 5.7, $S(\Gamma_2(x)) = S_{uaR(ab)}$. Now

$$S(\Gamma_2(x)) = S \cap G(\Gamma_2(x)) \geqq P_{uaM} = P(\Gamma_2(x)).$$

By orders, $S(\Gamma_2(x)) = P(\Gamma_2(x))$. But $S(\Gamma_2(x))$ fixes $R(ab)$ while $P(\Gamma_2(x))$ fixes M with $d(R(ab)), M) = 6$, a contradiction. Hence $P_{ua} = P(\Gamma_2(x))$. Dually, $P_{MN} = P(\Gamma_2(L))$.

By Lemma 5.2, $|P_{ua} : P_{uaM}| = p$ and so $P_{uaM} = P_{MNa}$ has order p. Furthermore

$P_{MNa} = P_{MN} \cap P_{ua} \trianglelefteq P$ and so $Z = Z(P) \cap P_{MNa} \neq 1$ and $Z = P_{MNa}$. It follows that $P_{ua}P_{MN} = P_{uN}$, a group of order p^3.

LEMMA 5.11. *P_{MN} has a set of $p+1$ distinct subgroups of order p, namely $\{P_{MN}(\Gamma_2(y)) : y \in L\}$.*

PROOF. If $y \in L$ and $z \in \Gamma_2(y) - L$, then P_{MN} fixes $R(yz)$ setwise since $R(yz) \in \Gamma_2(L)$ and so $|P_{MN} : P_{MNz}| = 1$ or p. If $P_{MN} = P_{MNz}$, then P_{MNa} fixes a and z with $d(a, z) = 6$, a contradiction. So P_{MNz} has order p, fixes $R(yz) - \{y, z\}$ setwise and so fixes $\Gamma_2(y)$ pointwise. Indeed

$$P_{MNz} = P_{MN} \cap P_{xz} = P_{MN} \cap P(\Gamma_2(y))$$

by Lemma 5.6. If $y' \in L - \{y\}$ and $z' \in \Gamma_2(y') - L'$, then $P_{MNz} \neq P_{MNz'}$ because $d(z, z') = 6$. This completes the proof of the lemma.

Now $P_{MN} = P(\Gamma_2(L))$ is a Sylow p-subgroup of $G(\Gamma_2(L) \cup \{L\})$. By Lemma 4.1, G_L is doubly transitive on the points of L. Hence by the Frattini argument, $N(P_{MN})_L$ is doubly transitive on the set of $p+1$ subgroups and induces at least $SL(2, p)$ on P_{MN}, because

$$G_L = N(P_{MN})_L \cdot G(\Gamma_2(L) \cup \{L\}).$$

LEMMA 5.12. *P_{ua} has a set of $p+1$ distinct subgroups of order p, namely $\{P_{ua}(\Gamma_2(R)) : R$ is a line on $x\}$.*

The proof is the dual of the previous proof and is omitted.

By Lemma 4.3, G_x is doubly transitive on the singular lines through x. The group $P_{ua} = P(\Gamma_2(x))$ is a Sylow p-subgroup of $G(\Gamma_2(x) \cup \{x\})$. By the Frattini argument, $N(P_{ua})_x$ is doubly transitive on the set of $p+1$ subgroups of P_{ua} and induces at least $SL(2, p)$ on P_{ua}.

In view of the action of $N(P_{ua})_x$ on P_{ua}, there is a 2-element $t \in N(P_{ua})_x \cap N(P_{MN})$ which inverts P_{ua} and centralizes P_{MN}/Z. Then t normalizes each of the $p+1$ subgroups of P_{ua} corresponding to the lines on x and so $t \in G(\Gamma_1(x))$. Similarly there is a 2-element $t' \neq t$ with $t' \in N(P_{MN})_L \cap N(P_{ua})$ which inverts P_{MN} and centralizes P_{ua}/Z. Then $t' \in G(\Gamma_1(L))$. We assume that $\langle t, t' \rangle \leqq N(P_{uN})$ is a 2-group.

Now tt' centralizes Z and inverts P_{uN}/Z. Hence $tt' \in G(\Gamma_1(x)) \cap G(\Gamma_1(L))$. For $y \in L - \{x\}$, the element tt' fixes $\Gamma_1(y)$ and so fixes one of the p lines $\neq L$ on y, say L_1. Since tt' fixes $L_1 - \{y\}$, it fixes one of the p points of $L_1 - \{y\}$. Now $Z = P_{MNa} \leqq P(\Gamma_1(y))$ and is transitive on $L_1 - \{y\}$, lest Z fix a and i with $d(a, i) = 6$. Since tt' centralizes Z, it follows that $tt' \in G(\Gamma_1(L_1))$. Similarly for

$u \in L - \{x, y\}$, the element $tt' \in G(\Gamma_1(L_2))$ for some $L_2 \neq L$ on u. Let $i \in L_1 - \{y\}$ and $j \in L_2 - \{u\}$. Since tt' fixes $\Gamma_1(j)$, it fixes one of the p lines $\neq L_1$ on j, say K. Because $d(j, i) = 6$, it follows that $d(K, i) = 5$. Let K, m_1, K_1, m_2, K_2, i be the unique path of $\mathscr{H}$ joining K to i. Then tt' must fix this path. Otherwise the vertices i, m_2, m_1, $tt'(m_1)$, $tt'(m_2)$ form either a quadrangle or a pentagon of $\mathscr{H}$ since tt' fixes the line K and the vertex i, a contradiction. Thus tt' fixes the vertices of the hexagon y, u, j, m_1, m_2, i. In particular tt' fixes $L = R(uy)$ and $K_1 = R(m_1m_2)$ with $d(L, K_1) = 6$. Since tt' fixes all points of L and all lines on $x \in L$, it now follows by Lemma 4.6 that $tt' = 1$. This contradiction completes the proof of Theorem 1.2.

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